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A Duality Concept for the Analysis of Polyhedral Scenes*

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INTRODUCTION

Pictures of scenes containing polyhedra do not record directly the orientations of the various plane bounding surfaces nor the exact locations of the edges and vertices of the objects that are portrayed. An important goal of the analysis of such pictures is to obtain as much information as is possible about these orientations and locations and to record it in a manner that is as easy to understand as is possible. The author introduced a simple means of visualizing this information in an earlier paper (Huffman, 1971). That paper dealt with procedures that could assist one in deciding whether the objects portrayed in certain idealized pictures were realizable or not. The representation that was referred to as a "dual picture-graph" was found to be quite useful as a tool in making these decisions. A more general and powerful representation than the one reported earlier has since been found by the author to be even more useful; for example in a paper ("Realizable configurations of lines in pictures of polyhedra") to be found elsewhere in these proceedings.

The newer representation is an elaboration of the older one and requires a three-dimensional space. Specifically, each member of a given family of parallel planes in the space of the scene corresponds to a unique point in the space of the dual-scene, each line in the scene corresponds uniquely to a line in the dual-scene, and each point uniquely to a plane. The duality concept proposed here is completely symmetric. That is, the mapping of planes, lines, and points of the scene onto the points, lines, and planes of the dual-scene is exactly the same as the one that maps these same entities in the dual-scene onto their correspondents in the scene.

If orthographic projections are made of the objects in the (x,y,z) coordinate system of the *scene* and of the corresponding objects in the (u,v,w) coordinate system of the dual-scene, the *picture* and dual-picture, in the (x,y) and (u,v)

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coordinate systems, respectively, result. These two pictures have precisely complementary roles: the orientations of planes and slopes of lines, that are not directly recorded in one representation, are recorded in the dual representation, and vice versa.

Several examples of the use of the dual-scene are included in this paper. Finally, an extension of the duality concept that is appropriate for more general surfaces in reported briefly.

DEFINITION OF THE DUAL OF A PLANE

We distinguish between the three dimensional space of the *scene* ((x,y,z) coordinates) and the three dimensional space of the *dual-scene* ((u,v,w) coordinates). Similarly we distinguish between the two dimensional space of the *picture* ((x,y) coordinates) and the two dimensional space of the *dual-picture* ((u,v) coordinates) that are obtained by orthographic projection. Thus, in the picture information about z values is unrecorded and in the dual-picture information about w values is unrecorded. Furthermore, we adopt the convention that z (and w) increase in the direction away from the viewer. Because we have assumed that pictures and dual-pictures are obtained by orthographic projection there is no loss of generality in taking the plane z=0 to be the picture plane and the plane w=0 to be the dual-picture plane.

The entity dual to the plane z=ax+by+c in the scene is defined to be the point at u=a, v=b, w=-c in the dual-scene. As a consequence all planes parallel to a given plane have common values of the parameters a and b, but differing values of the parameter c. The corresponding set of distinct points in the dual-scene all project onto the same point (u=a, v=b) in the dual-picture. It also follows that the infinite set of planes having the point x=0, y=0, z=c in common maps onto the infinite set of points contained in the plane w=-c.

In the dual-picture the distance from the origin to the point u=a, v=b equals the tangent of the angle between the corresponding scene plane and the plane z=0. Furthermore, the direction from the origin to the point u=a, v=b is indicative of the direction in which the scene plane tilts. For example, when a is positive and b is negative the value of z on the plane in the scene will increase (corresponding to an increase in range from the viewer) when x is increased and y is decreased.

Alternately, the point at u=a, v=b (and w=0) in the dual-picture can be thought of as the point of intersection between the plane w=0 and a line passing through the point u=0, v=0, w=1 when that latter line has the direction that is normal to the family of scene planes represented by the given point. (Note also the discussion associated with Figure 9, following.)

THE DUALS OF SCENE LINES AND SCENE POINTS

Consider the intersection of the two planes defined by

$$z = a_1 x + b_1 y + c_1$$
$$\text{and } z = a_2 x + b_2 y + c_2$$

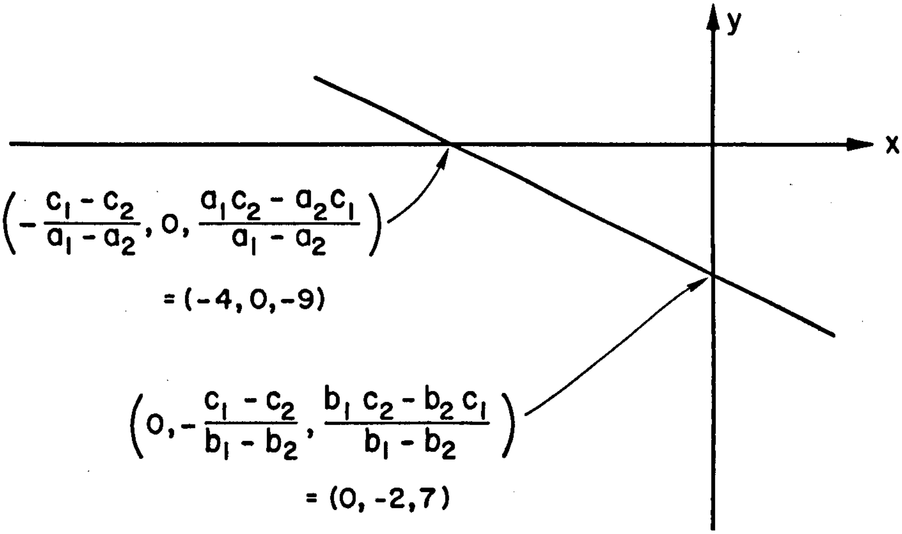


FIG. 1. A picture line.

The resulting line satisfies the equations

$$
\begin{aligned}
0 &= (a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) \\
(a_1 - a_2)z &= (a_1b_2 - b_1a_2)y + (a_1c_2 - c_1a_2) \\
(b_1 - b_2)z &= (b_1a_2 - a_1b_2)x + (b_1c_2 - c_1b_2).
\end{aligned}
$$

An example of this intersection line, drawn to scale for the case $a_1=3, b_1=-2, c_1=3, a_2=2, b_2=-4, c_2=-1$ is shown in the picture of Figure 1. We note for future reference that the length of the *picture line* contained between the x-axis and y-axis intercepts is

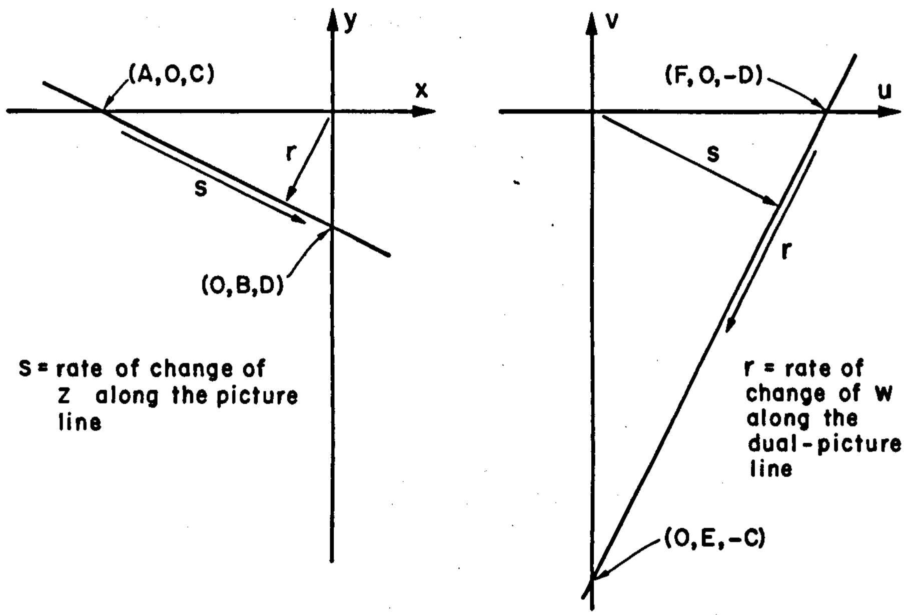
$$
\left| \frac{(c_1 - c_2)\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}}{(a_1 - a_2)(b_1 - b_2)} \right|
$$

This length is $2\sqrt{5}$ for our example.

Also, the change in z between the corresponding two points on the *scene line* has the magnitude

$$
\left| \frac{(c_1 - c_2)(a_1b_2 - b_1a_2)}{(a_1 - a_2)(b_1 - b_2)} \right|
$$

This change in z is 16 for our example.



(a) the picture

(b) the dual-picture

$$AF = -BE = C - D$$

$$r = \frac{AB}{\sqrt{A^2 + B^2}} = \frac{D - C}{\sqrt{E^2 + F^2}}$$

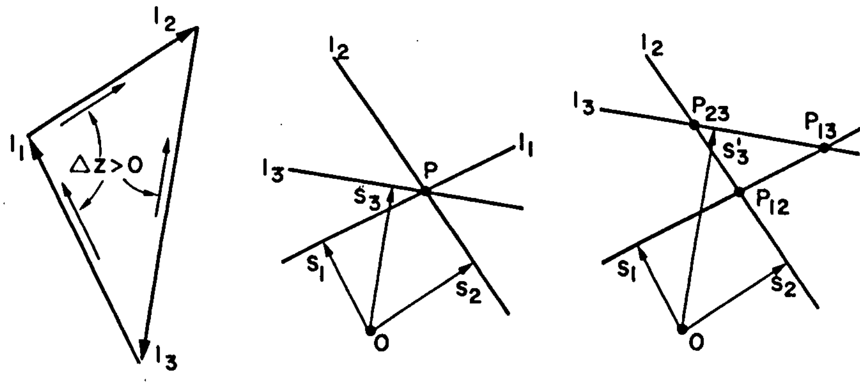
$$s = \frac{-EF}{\sqrt{E^2 + F^2}} = \frac{D - C}{\sqrt{A^2 + B^2}}$$

(c) relationships among line parameters

FIG. 3. Illustrating relationships between a scene line and a dual—scene line.

set of three planes in the scene has the defined dual relationship to the plane determined by the three corresponding points in the dual-scene. Therefore the duality is complete: plane to point, line to line, and point to plane.

To a given line in the scene there corresponds a uniquely determined line in the dual-scene. The line in the dual-scene can be thought of as comprised of an infinite set of points each element of which represents one of the set of planes containing the given line in the scene. Alternately, the line in the dual-scene is contained in all those planes of the dual-scene that correspond to the infinite set of points that constitute the given line in the scene. Similarly, to a given point in the scene there corresponds a uniquely determined plane of the dual-scene. Each point of that plane corresponds in the scene to one of the infinite set of planes



(a) picture

(b) dual-picture associated with a closed path in the scene

(c) dual-picture associated with an open path in the scene

FIG. 4. The analysis of an apparently closed path in a picture.

that contain the given point. Because of the completely symmetric relationship between the scene and the dual-scene each of the statements above is valid when the words "scene" and "dual-scene" are interchanged.

CONDITIONS FOR CLOSURE OF A SEQUENCE OF LINE SEGMENTS IN THE SCENE

Consider the closed sequence of three oriented line segments (hereafter called simply "lines" when no ambiguity will result) shown in Figure 4-a. It is assumed that all of these lines lie in a common plane, P. In the dual-picture of Figure 4-b the corresponding three lines will all pass through the point P associated with the given plane. The placement of that point with respect to the origin of the dual picture tells us that in this example the plane P in the picture is tilted up and to the right. The distance from the origin to the point in the dual-picture equals the tangent of the angle of that tilt.

The "slopes" of the three lines in the picture are equal to the distances to the three lines in the dual-picture. Thus the total amount by which z changes in traversing the line ℓ_i is equal to the signed product $\ell_i s_i$ where ℓ_i and s_i are the length and slope of the line ℓ_i. (Where no confusion will result we shall use "ℓ_i" to refer both to the i^{th} line segment and to its length.) We observe that z increases as we move clockwise along the first and second lines and decreases as we transverse the third line. Because the net change in z around the closed sequence of picture lines must be zero it follows that $\sum_i \ell_i s_i = 0$. We also note that this result is independent of the position in the dual-picture of P relative to 0.

If we modify the dual-picture so that all three lines do not intersect at a

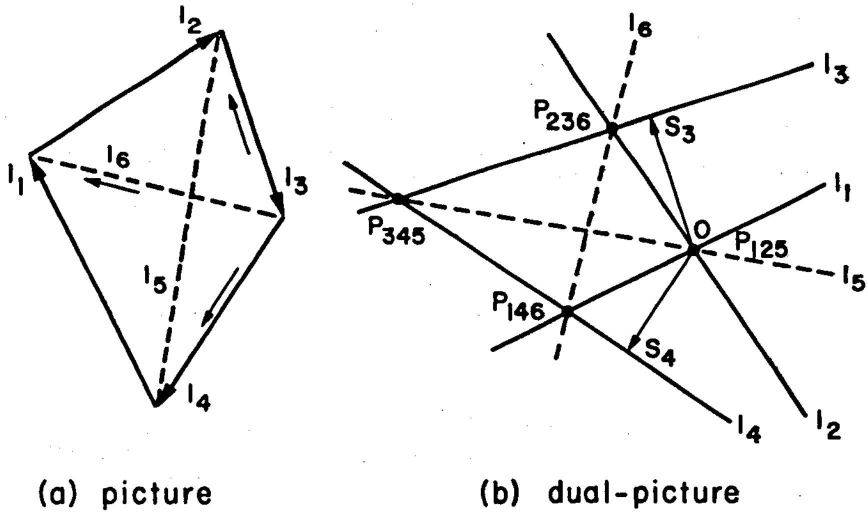


FIG. 5. The representation of a closed picture path.

common point (for instance, by moving ℓ_3 as shown in Figure 4-c) three different points are determined instead of only one and it becomes clear that the expression $\sum_i \ell_i s_i$ is no longer equal to zero. This means that the three lines that apparently form a closed path (in the picture) do not actually form a closed path in the scene itself. In our example the modification we have made corresponds to increasing the magnitude of the slope of the third picture line. If we were to assume, for instance, that the upper end of the third picture line remained fixed then at the lower end of that line we know that the first and third lines only appear to intersect. The point P_{13} in the dual-picture then represents all those planes that are parallel to both ℓ_1 and ℓ_3.

A more complicated example of an apparently closed path of four lines is shown in the picture of Figure 5-a. There are many possible ways of drawing the associated four lines in the dual-picture so that each is at right angles to the corresponding picture line. Not all of these give a set of slopes corresponding to a closed picture path. One that does is shown in Figure 5-b. If for convenience we choose the intersection of ℓ_1 and ℓ_2 to be the dual-picture origin we note that the net change in z around the picture path will be zero only if $s_3/s_4 = -\ell_4/\ell_3$. Thus the relative lengths of the line segments in the dual-picture are fixed if we want that representation to depict a closed picture path. It is easy to prove that when the appropriate proportions are present the line ℓ_5 in the picture will be at right angles to its correspondent in the dual-picture. A similar comment holds true for the line ℓ_6.

CONDITIONS AMONG THE LINES OF A CUT-SET

The examples of the previous section illustrate constraints in a dual-picture

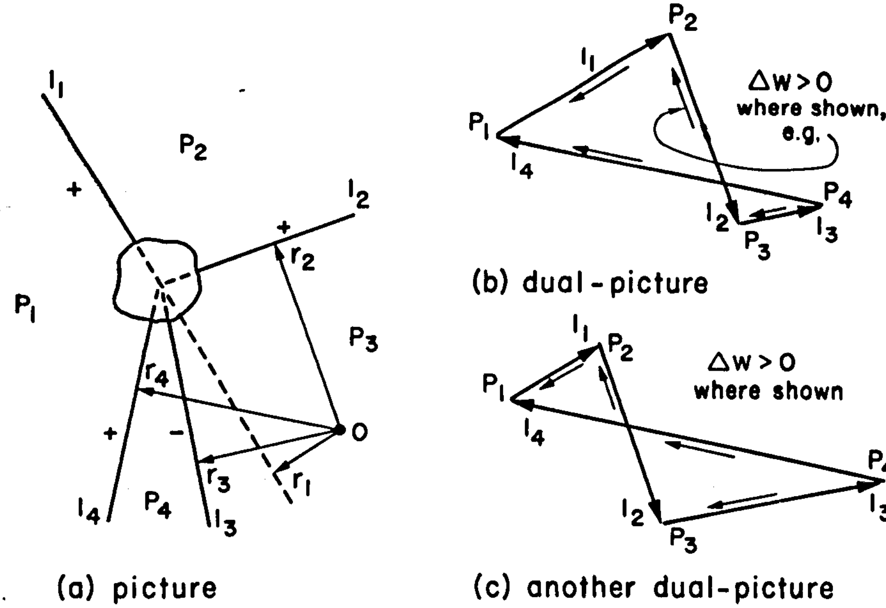


FIG. 6. A single-vertex cut-set.

that correspond to conditions for the closing of a *path* of lines in the scene. In this section we illustrate constraints that correspond to the topologically dual concept: a *cut-set* of lines. Here we consider a cut-set of picture lines to be all those lines that enter a simple closed region of the picture from outside that region. The region can be that around a single vertex and the lines of the cut-set those that depict all the (visible) edges incident at that vertex, as in Figure 6-a. More generally the region can include more than one vertex and the lines of the cut-set may have arbitrary positions, as is illustrated in Figure 7-a.

In Figure 6-a we assume that the four edges portrayed are incident at the single vertex. The line-labels indicate that two of the edges are "convex" (+) and one is "concave" (-). A clockwise encircling of the region associated with the cut-set corresponds in the dual-picture to a closed path along the directed lines of the dual representation. There are many different sets of four planes that could be present in the scene that would yield the given picture. For any one of these interpretations the corresponding points in the dual-scene are well-defined (and lie in a common plane). It necessarily follows that the net Δw around the closed path in the dual-scene is zero and, therefore, that $\sum_i \ell_i r_i = 0$. One possible choice of picture origin is shown in the figure. Two possible choices of orientations of the planes that are depicted in the picture would give the dual representations of Figure 6-b and 6-c.

We note that the issues raised in the example of Figure 6 are analogous to those raised in Figure 4. A closed path of lines and associated set of intersection points all contained in a single plane in one representation correspond in the

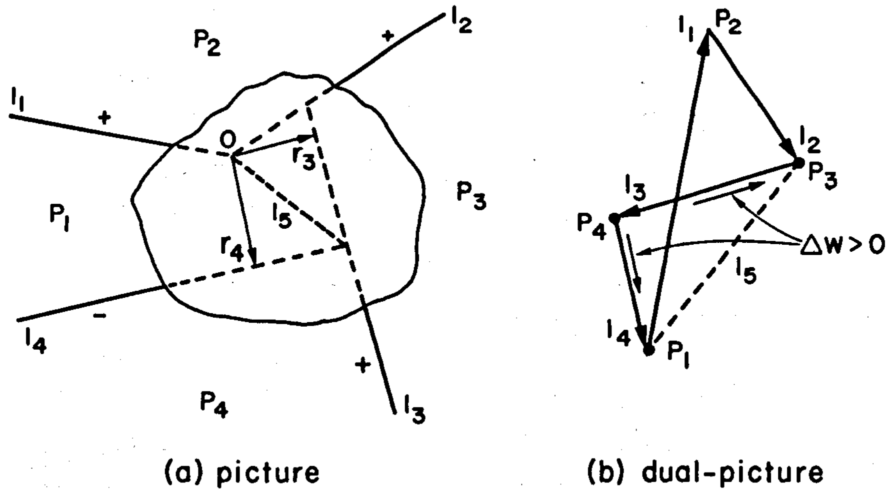


FIG. 7. A multi-vertex cut-set.

other representation to a set of lines and associated planes that all contain a single point.

The example shown in Figure 7 is more general than the previous one in that the four planes depicted in the picture cannot all contain a common vertex. The four lines of the dual-picture must, of course, be orthogonal to their correspondents in the picture. In this new example the ratios between pairs of lengths of dual-picture lines cannot be chosen arbitrarily. This can be easily seen if, for instance, we choose the picture origin to be the point shown. The reason for making this particular choice is that the distances from the origin to the third and fourth lines will then be zero. It is clear that then the ratio of the magnitudes of r_1 and r_2 is fixed. In order that $\sum_i \ell_i r_i = 0$ the ratio of the magnitudes of ℓ_1 and ℓ_2 in the dual-picture must be the one shown in the figure.

The dual-picture of Figure 7-b has the correct proportions. As a verification of this contention we note that one possible completion of the picture is that shown by dotted lines. The fifth line (common to planes P_1 and P_3 in the picture) is perpendicular, as is required, to its correspondent in the dual-picture. The points a and b in the scene correspond to the two planes determined by P_1, P_2, and P_3 and by P_1, P_3, and P_4, respectively, in the dual-scene.

The issues raised in the examples of Figures 5 and 7 are analogous. In each case in one representation a cyclic sequence of planes and the set of lines formed by the intersection of planes that are pairwise adjacent in that sequence correspond in the other representation to a cyclic sequence of points and the set of lines determined by points that are pairwise adjacent in that sequence.

AMBIGUITY OF SCALE AND ORIGIN

A picture inherently contains two kinds of ambiguity. Information about the

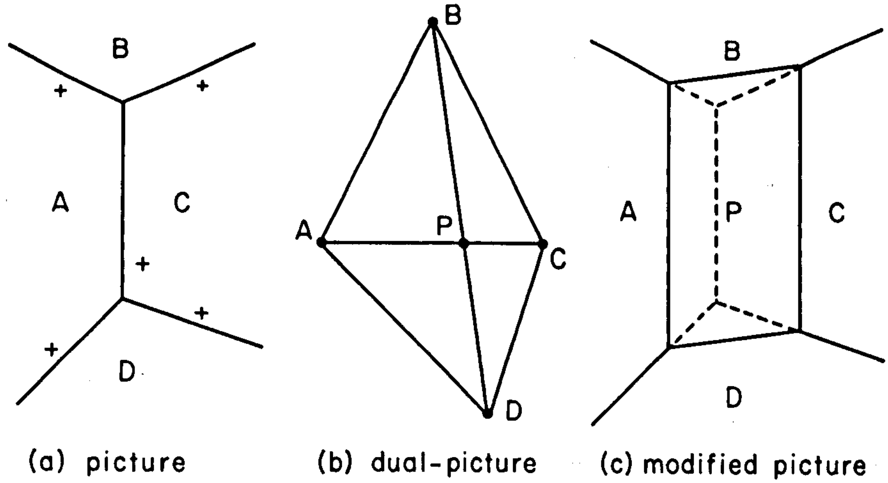


FIG. 8. An origin- and scale-independent problem.

depth of the object(s) depicted is lost because of the necessary projection. The visible planes of a polyhedral object may all have orientations that are nearly alike; that is, the object may be "shallow". On the other hand the same picture may be of a "deep" object, in which case the directions in which the normals to the various planes point may widely differ. These two situations may be represented by two different dual-pictures. The shallow object would have a dual-picture in which the various points (associated with the directions taken by the normals to the scene planes) were all close together. The points in the dual-picture for a deeper object would be further apart.

By changing the scale of a dual-picture we can represent either of the situations above. For instance, multiplying the lengths of all lines in a dual-picture by a factor $k > 1$ corresponds to multiplying by k the slopes of all lines in the scene. Similarly, multiplying the lengths of all scene lines by k corresponds to multiplying the slope of all dual-scene lines by k. It is easy to see that a "size" change in an object in the scene would cause a proportionate change in the recorded picture but no effect in the dual-picture. A change in depth of the scene object would cause a change in the size of the dual-picture portrayal even though no change would be apparent in the picture.

Even when the sizes of the picture and dual-picture images are determined another kind of ambiguity still remains. The distance from the dual-picture origin to each point in the representation is associated with the (tangent of the) angle between the corresponding scene plane and the reference plane z=0. Each possible position of the dual-picture origin corresponds to a different set of orientations of the scene planes. And yet each such possibility can correspond to the given picture. Similarly, a change in position of the picture origin is associated with a change in the orientations of the planes of the dual-scene.

A valuable feature of the picture/dual-picture pair of representations in

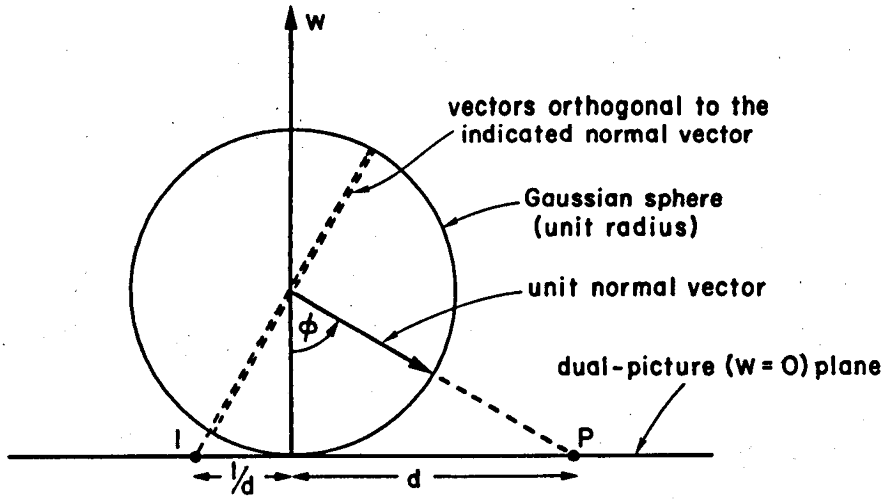


FIG. 9. Representation of planes orthogonal to a given plane.

dealing with many analysis problems is that often we need not concern ourselves with the scale or placement of the origin in either the picture or dual-picture. Consider for instance the (partially complete) picture of Figure 8-a. All five lines have labels indicating that the edges of the object are convex toward the viewer. How can we slice through the object with a plane P so that the new surface that results has the shape of a parallelogram?

We find that orientations of the lines of the dual-picture are determined even though its origin and scale are not specified (see Figure 8-b). If the line common to A and P and the line common to C and P are to be parallel (in the scene and in the picture) the corresponding dual-picture lines must be parallel or must be the same line. Similarly, the lines common to B and P and to D and P will be parallel or the same line when the corresponding dual-picture lines are parallel. Clearly the point P shown in the dual-picture is the only possible one. By noting the orientation of the line BPD we determine (by perpendicularity) the orientation of the lines common to B and P and to D and P (see Figure 8-c).

The size of the parallelogram is not determined by the problem statement. Any of an infinite number of planes parallel to P would also give parallelogram slices through the object. All such planes are represented by the same dual-picture point.

REPRESENTATION OF ORTHOGONAL PLANES

There are many situations in which it is important to be able to represent the set of those planes that are orthogonal to a given plane. In this section we will discuss this representation and give examples of its use.

Consider a given plane P in the scene. Let the direction of the normal vector be that shown in Figure 9. Let the base of that normal be at the point u=v=0,

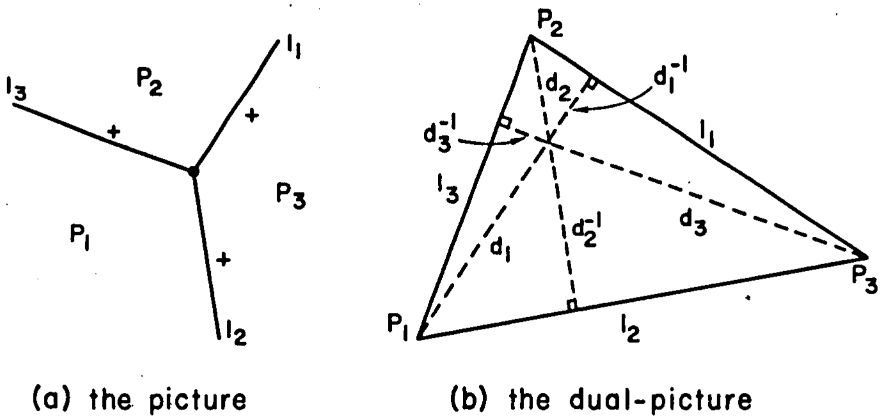


FIG. 10. Representation of a cube vertex.

w=1 and its intersection with the w=0 plane be at u=a, v=b at distance $d = \sqrt{a^2 + b^2}$ from the origin. The angle ϕ between the plane P and the picture plane (z=0) is the angle between the normal vector described above and the w-axis. As we have commented earlier $d = \tan \phi$.

It is convenient in dealing with normal vectors to consider the unit radius "Gaussian sphere" and the end points of those vectors on that sphere (Hilbert and Cohn-Vossen, 1952). In this case the normals to planes orthogonal to P are represented on a great circle of the sphere 90° away from the point associated with P. The corresponding points in the dual-picture plane are found on the line ℓ, the end-view of which is shown in the figure. The distance from the origin to this line is $1/d$.

An example of the use of representations of orthogonal planes is given in Figure 10. The picture given is assumed to be that of a cube vertex. The dual-picture has the constraints that are indicated in Figure 10-b. In particular note that in that representation a line ℓ_i represents all those planes that are orthogonal to plane P_i. The three dotted lines through the origin are perpendicular to ℓ_1, ℓ_2, and ℓ_3 in the dual-picture and therefore parallel to their correspondents in the picture. It is apparent that if the picture of the edges of a cube is given and if the orientation of one plane (say P_1) in the scene is given the orientations of the other two planes is uniquely determined. This fact is apparent in the dual-picture as well.

Another example of the representation of orthogonal planes is given in Figure 11. The problem posed is as follows. In the picture two lines ℓ_1 and ℓ_2 are given, each with a specified slope. The separation (change in the value of z) at their apparent intersection is also given. The picture of their common perpendicular is to be determined.

The specification of the slopes s_1 and s_2 determines ℓ_1 and ℓ_2 in the dual-picture. Their intersection point P corresponds to the set of planes that are parallel to both ℓ_1 and ℓ_2 in the picture. The desired perpendicular in the scene is

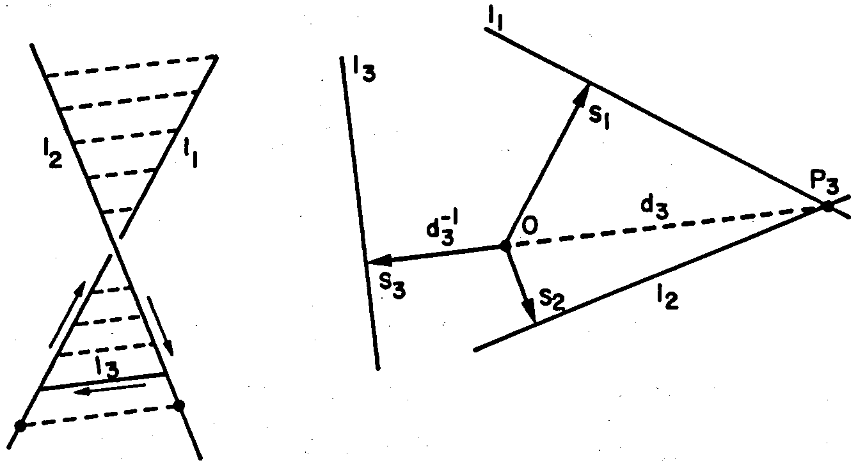


FIG. 11. Determination of a common perpendicular.

a line common to a set of planes all of which are orthogonal to the set corresponding to P. Thus the line ℓ_3 in the dual picture represents this perpendicular. Its slope is determined by the specifications of the problem. All of the dotted lines in the picture have a direction consistent with that of l_3 in the dual-picture. It is apparent that since the slopes of ℓ_1, ℓ_2, and ℓ_3 are now all determined there is a unique choice of ℓ_3 in the picture that will satisfy the given change of z at the apparent intersection of ℓ_1 and ℓ_2.

THE DETERMINATION OF SHADOWS

Another application of the representation of orthogonal planes is in the determination of shadows. Consider the dual-picture of Figure 12. The point S represents the "sun" (assumed to be infinitely distant from the objects in the scene). The distance d is the tangent of the angle between the vector pointing to the sun and the normal to the reference plane. For instance, if d were equal to zero the sun would be directly overhead for an observer standing on the reference plane. The line ℓ_s represents all those planes that have normals that are at right angles to the vector pointing to the sun. Planes such as those corresponding to the points A and B that are on one side of ℓ_s have normals with angles less than 90° from the sun vector. Planes such as C and D on the other side of ℓ_s have associated angles that exceed 90°. Therefore A and B are oriented so that the sun illuminates them and C and D are shadowed. For an observer standing on a plane such as E that is represented by a point on ℓ_s the sun would appear to be on the horizon.

Consider now the simple configuration of three planes and associated edges depicted in Figure 13-a. The labels on the lines indicate a (convex) "crest" line shared by planes B and C and two other (concave) lines where those two planes

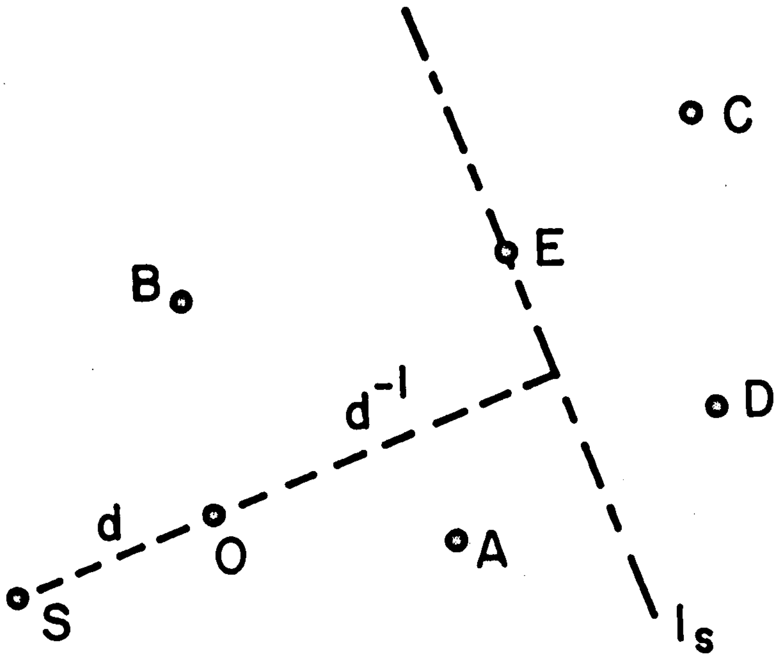


FIG. 12. Representation of the "sun" in the dual–picture.

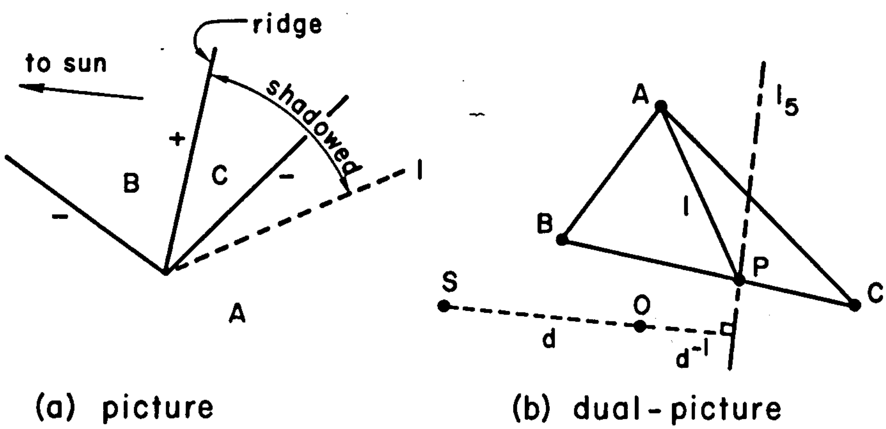


FIG. 13. Determination of the shadow of a line on a plane.

intersect the plane A. We assume that the orientations of these planes correspond to the points A, B, and C shown in the dual-picture. We also assume that the sun vector is the one shown. The line ℓ_s is constructed as indicated. Since S and C are on opposite sides of that line the plane C is turned away from the sun. It is apparent that under these conditions the crest-line will cast a shadow on plane A.

In order to determine the shadow-line ℓ cast by the ridge we first observe that

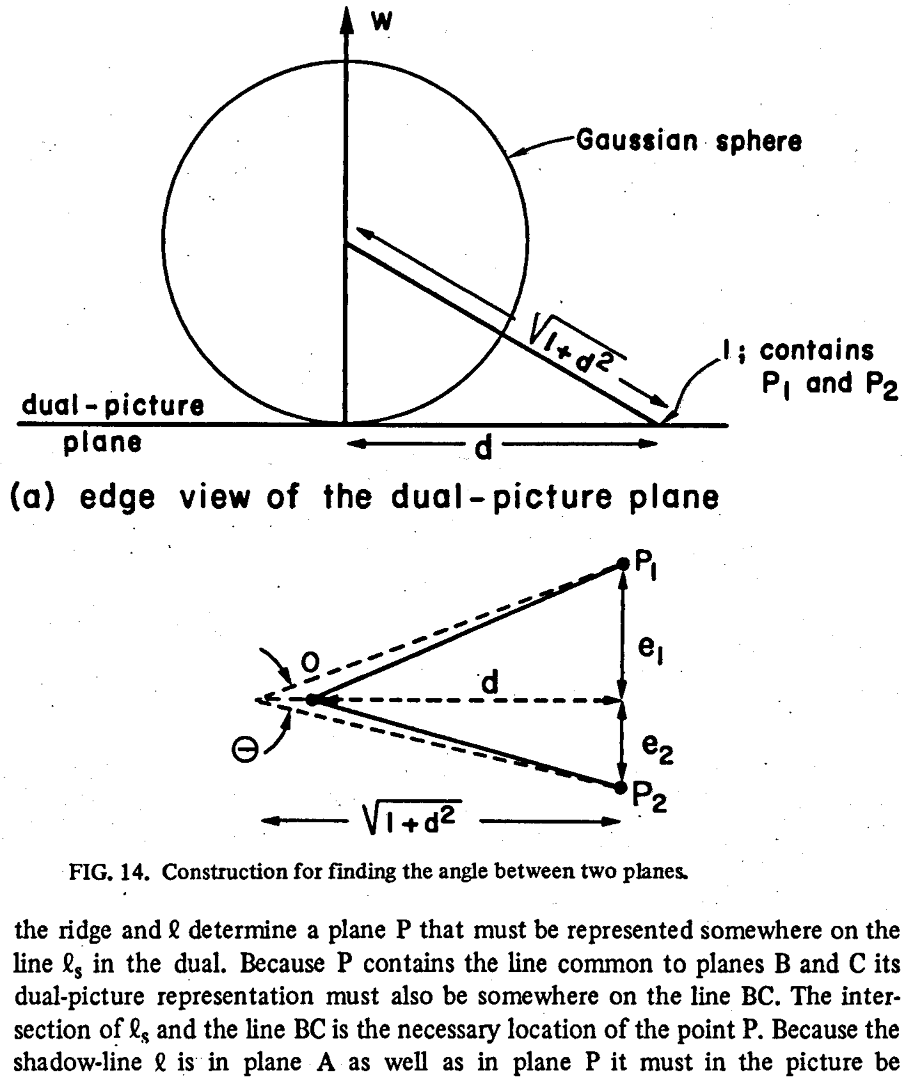


FIG. 14. Construction for finding the angle between two planes.

the ridge and ℓ determine a plane P that must be represented somewhere on the line ℓ_s in the dual. Because P contains the line common to planes B and C its dual-picture representation must also be somewhere on the line BC. The intersection of ℓ_s and the line BC is the necessary location of the point P. Because the shadow-line ℓ is in plane A as well as in plane P it must in the picture be perpendicular to the corresponding dual-picture line that contains the points A and P.

DETERMINATION OF THE ANGLE BETWEEN TWO PLANES

It is convenient to be able to find the angle between two planes from their representations in the dual-picture. The construction shown in Figure 14 illustrates one method of achieving this result. In Figure 14-b the representations for two planes P_1 and P_2 are given. A perpendicular from the origin to the line ℓ containing P_1 and P_2 has a length d and divides that line into two segments of lengths e_1 and e_2. If we were to look at the dual-picture edgewise and from the direction parallel to ℓ the resulting view would be that given in Figure 14-a.

The true angle between the two planes is the angle between their normals. This angle is that between P_1 and P_2 (in the dual-picture plane) as seen from the center of the Gaussian sphere. The distance to ℓ from the sphere center is $\sqrt{1+d^2}$. Therefore the construction shown in dotted lines in the dual-picture gives the true angle, θ, between the two planes.

The angle θ can also be found from the expression

$$\theta = \tan^{-1} \frac{e_1}{\sqrt{1+d^2}} + \tan^{-1} \frac{e_2}{\sqrt{1+d^2}}$$

If P_1 and P_2 lie on the same side of the perpendicular to ℓ from 0 the values of e_1 and e_2 will have opposite signs and the expression must be evaluated accordingly.

BRIEF SUMMARY AND A GENERALIZATION OF THE DUAL-SURFACE CONCEPT

The preceding sections of this paper have dealt with a concept of duality that was introduced for the purpose of analyzing pictures of scenes containing plane-bounded objects (polyhedra). This new concept is one that treats the "third" dimension of the scene or of the dual-scene in a manner that is distinct from the manner in which the other two dimensions are treated. The resulting picture and dual-picture pairs contain complementary types of information in the sense that depth information missing in one representation is present in the other representation, and vice versa. A generalization of this duality concept is reported briefly below. The author will elaborate on this generalization in a future paper.

We consider two surfaces defined by

$$z = f(x,y) \text{ and } w = g(u,v)$$

to be dual if for each point and associated tangent plane on one there is a corresponding tangent plane and associated point on the other. For the point (x_o,y_o,z_o) on the first surface we define the corresponding tangent plane to be $w = x_o u + y_o v - z_o$ on the second. Similarly for the point (u_o,v_o,w_o) on this second surface we define the corresponding tangent plane to be $z = u_o x + v_o y - w_o$ on the first surface. Thus for the pair of points of interest we have $w_o + z_o = x_o u_o + y_o v_o$. The point on one surface will be associated with the direction of the normal to the tangent plane on the other when

$$u_o = \left.\frac{\partial f}{\partial x}\right]_{x_o,\,y_o} \qquad v_o = \left.\frac{\partial f}{\partial y}\right]_{x_o,\,y_o}$$

$$x_o = \left.\frac{\partial g}{\partial u}\right]_{u_o,\,v_o} \qquad y_o = \left.\frac{\partial g}{\partial v}\right]_{u_o,\,v_o}$$

As a simple example consider the surface $z = f(x,y) = x^3 + xy$. We have

$$u = \frac{\partial f}{\partial x} = 3x^2 + y$$

$$v = \frac{\partial f}{\partial y} = x$$

These equations have the solution

$$x = v, \; y = u - 3v^2$$

Therefore

$$\begin{aligned} w &= ux + vy - z \\ &= (3x^2 + y)\,x + xy - (x^3 + xy) \\ &= 2x^3 + xy \\ &= 2v^3 + v(u - 3v^2) \\ &= -v^3 + uv \end{aligned}$$

Therefore the surface that has the desired dual relationship is given by $w = g(u,v) = uv - v^3$.

Finally, we observe that it is possible for a surface to be *self-dual*; that is, for the functions f and g to be the same. As an example, the reader can verify that

$$z = f(x,y) = \frac{(x^2 - y^2)\cos\theta \pm 2xy\sin\theta}{2}$$

describes a self-dual surface; in this case a saddle surface. The corresponding surface in the other coordinate system is

$$w = g(u,v) = \frac{(u^2 - v^2)\cos\theta \pm 2\,uv\sin\theta}{2}$$

The self-duality is independent of the value of the parameter θ.

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